$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \qquad \psi = \psi_x(x)\psi_y(y) \Rightarrow$$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\psi_x\psi_y = E\psi_x\psi_y$$

$$-\frac{\hbar^2}{2m}\psi_x''\psi_y - \frac{\hbar^2}{2m}\psi_x\psi_y'' = E\psi_x\psi_y$$

$$\underbrace{-\frac{\hbar^2}{2m}\frac{\psi_x''}{\psi_x}}_{E_x} \underbrace{-\frac{\hbar^2}{2m}\frac{\psi_y''}{\psi_y}}_{E_y} = E \quad \Rightarrow E_x + E_y = E$$

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{\psi_x''}{\psi_x} = E_x \Rightarrow \psi_x'' = -\frac{\overbrace{2mE_x}^{k_x^2}}{\hbar^2}\psi_x \to \psi_x = Ae^{ik_x x} + Be^{-ik_x x} \\ -\frac{\hbar^2}{2m}\frac{\psi_y''}{\psi_y} = E_y \Rightarrow \psi_y'' = -\frac{\overbrace{2mE_y}^{k_y^2}}{\hbar^2}\psi_y \to \psi_y = ce^{ik_y y} + De^{-ik_y y} \end{cases}$$

<u>Apply B.C.</u>

$$\begin{cases} \psi(x=0,y)=0 \to A+B=0 \to A=-B \to \psi_x = 2iA\sin k_x x \\ \psi(x,y=0)=0 \to C+D=0 \to C=-D \to \psi_y = 2iC\sin k_y y \end{cases}$$

$$\begin{cases} \psi(x=L,y)=0 \to \sin k_x L = 0 \to k_x = \frac{n\pi}{L} \quad n=1,2,\dots \\ \psi(x,y=L)=0 \to \sin k_y L = 0 \to k_y = \frac{m\pi}{L} \quad m=1,2,\dots \end{cases}$$

$$\Rightarrow \psi(x,y) = \psi_x\psi_y = A\sin k_x x \sin k_y y$$

Normalize: $\int_0^L dx \int_0^L dy\, |\psi|^2 = 1 \to A^2 \underbrace{\int_0^L \sin^2 k_x x\, dx}_{\frac{1}{2k_x}(k_x L - \underbrace{\sin k_x L}_{=0} \cos k_x L) = \frac{L}{2}} \int_0^L \sin^2 k_y y\, dy = 1$

$$\Rightarrow \quad A^2\left(\frac{L}{2}+\frac{L}{2}\right)=1 \rightarrow A=\frac{1}{\sqrt{L}}$$

So the eigenfunctions are:

$$\boxed{\psi_{nm}(x,y)=\frac{1}{\sqrt{L}}\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{L}x \qquad \begin{matrix} n=1,2,\dots \\ m=1,2,\dots \end{matrix}}$$

And the eigenvalues are:

$$\Rightarrow \quad E=E_x+E_y=\frac{\hbar^2k_x^2}{2m}+\frac{\hbar^2k_y^2}{2m}$$

$$\boxed{E_{nm}=\frac{\hbar^2\pi^2}{2mL^2}(n^2+m^2)}$$

For the ground state $n=m=1$:

$$E_{11}=\frac{\hbar^2\pi^2}{mL^2} \qquad \psi_{11}=\frac{1}{\sqrt{L}}\sin\frac{\pi}{L}x\,\sin\frac{\pi}{L}y$$

Nondegenerate and has odd parity

For the 1st excited state $n=1, m=2$ or $n=2, m=1$:

$$E_{12}=E_{21}=\frac{\hbar^2\pi^2}{2mL^2}(1^2+2^2)=\frac{5\hbar^2\pi^2}{2mL^2} \qquad \psi_{12}=\frac{1}{\sqrt{L}}\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L}$$

$$\psi_{21}=\frac{1}{\sqrt{L}}\sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}$$

Two fold degenerate & odd parity.